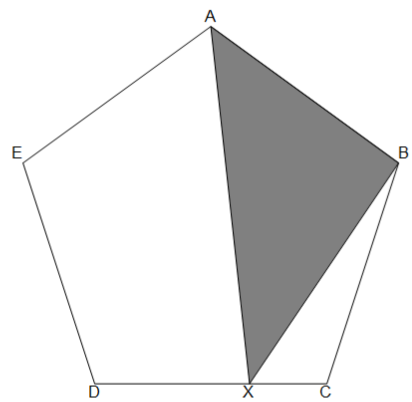
KENNESAW STATE UNIVERSITY
COLLEGE OF SCIENCE AND MATHEMATICS
Department of Mathematics

THE 2024–2025 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

Calculators are NOT permitted

Time allowed: 2 hours

1. **Problem.** ABCDE is a regular pentagon with side length 3. Point X is chosen on CD so that CX = 1 and XD = 2. Find, with proof, the fraction of the pentagon's area that is shaded.



2. **Problem.** Let *S* be the sum of the squares of the first 1000 odd primes. Prove that *S* is divisible by 24.

3. **Problem.** Find all real solutions to the equation

$$(x^2 - 2026 \cdot 2027)^2 - 2026 \cdot 2027 = x$$

4. **Problem.** Is there any positive integer *n* for which the expression

$$\frac{1 \cdot 2}{3} \cdot \frac{3 \cdot 4}{3} \cdot \frac{5 \cdot 6}{3} \cdot \frac{7 \cdot 8}{3} \cdot \frac{9 \cdot 10}{3} \cdot \ldots \cdot \frac{(2n-1)(2n)}{3}$$

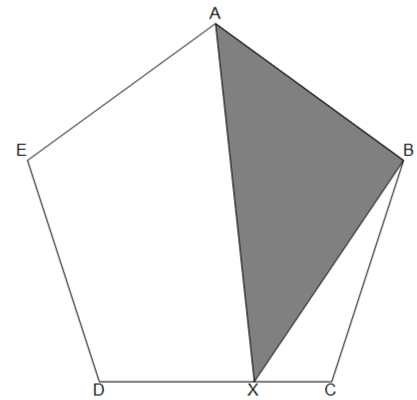
is an integer? Find the smallest value of *n* for which this happens or prove that it never does.

5. **Problem.** Pete and Cora are playing a game. First, they draw a circle. On each turn, Pete picks two points on the circle, and Cora draws the chord connecting them, coloring it either red or blue. After 10 turns, Pete wins if he can get Cora to either draw a red quadrilateral (whose sides don't intersect) or two blue intersecting chords (with four different endpoints); if neither of those is present, Cora wins.

   Assuming each player makes the optimum choice on each turn, determine with proof who will win.

Solutions

1. **Solution.** The answer is ⅓.

    We find the fraction of the area of the pentagon, which is unshaded, instead. This can be divided into three parts:

    - The area of triangle ADE;
    - The area of triangle AXD, which is ⅔ of the area of triangle ACD, because the two share the same height, but the base of AXD is 2 and the base of ACD is 3.
    - The area of triangle BCX, which is ⅓ of the area of triangle BCD, because the two share the same height, but the base of BCX is 1 and the base of BCD is 3.

    However, triangles BCD and ADE have the same area as each other, and both have the same area as triangle ABC. Therefore

    (area of ADE) + ⅓ (area of BCD) + ⅔ (area of ACD) =
    ⅔ (area of ADE) + ⅔ (area of BCD) + ⅔ (area of ACD) =
    ⅔ (area of ADE) + ⅔ (area of ABC) + ⅔ (area of ACD)

    and this makes up ⅔ of the area of the entire pentagon. Since ⅔ of the pentagon is unshaded, ⅓ is shaded.

2. **Solution.** Let's look at divisibility by 3 first.

    When we compute S, we start at $3^2 = 9$, which is divisible by 3. Every odd prime after that can be written as $3k+1$ or $3k-1$ for some k. We have $(3k+1)^2 = 9k^2 + 6k + 1$ and $(3k-1)^2 = 9k^2 - 6k + 1$; the first two terms are divisible by 3 in each case, so we can ignore them; with every prime square we add, the remainder when the sum is divided by 3 increases by 1. We add 999 more prime squares to S, and 999 is divisible by 3, so S is also divisible by 3.

    Now let's look at divisibility by 8. Every odd prime can be written as $2k+1$ for some k, which squares to $(2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. For any k, either k or $k+1$ is even, so $k(k+1)$ is even, and $4k(k+1)$ is divisible by 8. Therefore, every prime square is 1 more than a multiple of 8. We add together 1000 prime squares in S, and 1000 is divisible by 8, so S is also divisible by 8.

    Since S is divisible by both 3 and 8, it must be divisible by 24.

3. **Solution 1.** We can write this equation as $f(f(x)) = x$, where $f(x) = x^2 - 2026{\cdot}2027$. If $f(x) = x$, then it's certainly true that $f(f(x)) = x$, so two of our solutions can be found by solving the quadratic equation $x^2 - 2026{\cdot}2027 = x$. This equation factors as $(x - 2027)(x + 2026) = 0$, so its roots are -2026 and 2027.

There are four roots to the original quartic equation, so we still want to find two more. We can exclude the two roots we've found in a variety of ways, including synthetic division. Probably the cleanest is to expand $(x^2 - 2026{\cdot}2027)^2 - 2026{\cdot}2027 = x$ as

$$x^4 - 2{\cdot}2026{\cdot}2027x^2 - x + (2026^2{\cdot}2027^2 - 2026{\cdot}2027) = 0$$

and then observe that by Vieta's formulas, the sum of its roots must be 0 (the negative of the coefficient of x^3) and the product must be $2026^2{\cdot}2027^2 - 2026{\cdot}2027$ (the constant term). The two roots we've found already have sum 1 and product $-2026{\cdot}2027$, so the other two roots must have sum -1 and product $1 - 2026{\cdot}2027$: they must be the roots of the quadratic equation

$$x^2 + x + (1 - 2026{\cdot}2027) = 0.$$

From here, we can use the quadratic formula to find the final two roots:

$$x = \frac{-1 \pm \sqrt{1 - 4{\cdot}(1 - 2026{\cdot}2027)}}{2}.$$

(Since the discriminant $4{\cdot}2026{\cdot}2027-3$ is positive, the last two roots are also real.)

Solution 2. Adding $-x^2$ both sides and factoring:

$$(x^2 - 2026 \cdot 2027)^2 - 2026 \cdot 2027 - x^2 = x - x^2$$
$$(x^2 - 2026 \cdot 2027)^2 - x^2 = -(x^2 - x - 2026 \cdot 2027)$$
$$(x^2 - x - 2026 \cdot 2027)(x^2 + x - 2026 \cdot 2027) = -(x^2 - x - 2026 \cdot 2027)$$
$$(x^2 - x - 2026 \cdot 2027)(x^2 + x - 2026 \cdot 2027) + (x^2 - x - 2026 \cdot 2027) = 0$$
$$(x^2 - x - 2026 \cdot 2027)(x^2 + x - 2026 \cdot 2027 + 1) = 0$$
$$(x - 2027)(x + 2026)(x^2 + x - 2026 \cdot 2027 + 1) = 0$$

Setting each factor equals to zero we have:

$$x = -2026 \text{ , } x = 2027 \text{ , } x = \frac{-1 \pm \sqrt{1 - 4{\cdot}(1 - 2026{\cdot}2027)}}{2}$$

4. **Solution.** No, this is not possible.

Going up to *2n*, the number of multiples of 3 is 2*n*/3, rounded down. But that's not the total number of times that the numerator is divisible by 3: there are also 2*n*/9 (rounded down) multiples of 9, which provide an additional factor of 3, and 2*n*/27 (rounded down) multiples of 27, which provide a third factor of 3, and so on.

If we were to ignore the rounding, and extend this sum infinitely, we would get

$$\frac{2n}{3} + \frac{2n}{9} + \frac{2n}{27} + \frac{2n}{81} + \dots = \frac{2n/3}{1 - 1/3} = n$$

factors of 3 in the numerator. Since the terms do get rounded down, and the sum is not infinite (after a certain point, the number of multiples of 3^k in the numerator is just 0), the actual number of factors of 3 is strictly less than *n*.

On the other hand, there are exactly *n* factors of 3 in the denominator. Therefore, there are always more factors of 3 in the denominator than in the numerator, and the expression can never be an integer.

5. **Solution.** Pete has a winning strategy.

Here's one possible strategy. First, Pete asks Cora to draw all four sides of a quadrilateral; if all four sides are red, Pete has already won. Otherwise, one of the sides is blue; call that chord AB.

Next, Pete asks Cora to draw a pair of intersecting chords on one side of AB. If both are blue, that's enough for Pete to win, and otherwise, one is red; call it PQ. Next, Pete does the same thing on the other side of AB, either winning, or getting a red chord RS. Assume that the six points here are in order A, P, Q, B, R, S around the circle.

That's 8 turns already, but Pete can win in the last two. On the 9^{th} turn, Pete picks Q and R; on the 10^{th} turn, Pete picks P and S. If Cora colors either one of them blue, then that's a blue chord intersecting the blue chord AB, and Pete wins. If Cora colors both of them red, then PQRS is a red quadrilateral, and Pete also wins.