KENNESAW STATE UNIVERSITY
COLLEGE OF SCIENCE AND MATHEMATICS
Department of Mathematics

THE 2024–2025 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

1. **Problem.** On every twig of a tree, there are six leaves. On every branch of the tree, there are three twigs and three additional leaves. On every bough of the tree, there are two branches, two additional twigs, and two additional leaves. If the tree has six boughs, how many leaves does it have?

   (A) 216 (B) 300 (C) 324 (D) 336 (E) 354

2. **Problem.** Aaron, Barbara, and Cece bought the same kinds of flour and sugar from the same grocery store. Aaron bought twice as much flour as Barbara, but Barbara bought twice as much sugar as Aaron, and Aaron paid 25% more than Barbara. If Cece bought as much flour as Aaron and as much sugar as Barbara, how much more did Cece pay than Aaron?

   (A) 10% more (B) 15% more (C) 20% more (D) 25% more (E) Cannot be determined

3. **Problem.** Simplify the expression
$$\frac{123 + 132 + 213 + 231 + 312 + 321}{789 + 798 + 879 + 897 + 978 + 987}$$
   as much as possible.

   (A) $\frac{1}{4}$ (B) $\frac{2}{7}$ (C) $\frac{1}{3}$ (D) $\frac{4}{9}$ (E) $\frac{1}{2}$

4. **Problem.** Four siblings—Peter, Quinn, Reese, and Sofia—baked cookies together. Overnight, the cookies disappeared, and only one of the siblings could have taken them. When arguing about who it was, they had the following to say:

   Peter: Either Quinn or Reese took the cookies.
   Quinn: It wasn't me!
   Reese: Sofia didn't take the cookies.
   Sofia: I didn't take the cookies, and neither did Quinn.

   If the one who took the cookies lied, and the other three told the truth, then who took the cookies?

   (A) Peter (B) Quinn (C) Reese (D) Sofia (E) Cannot be determined

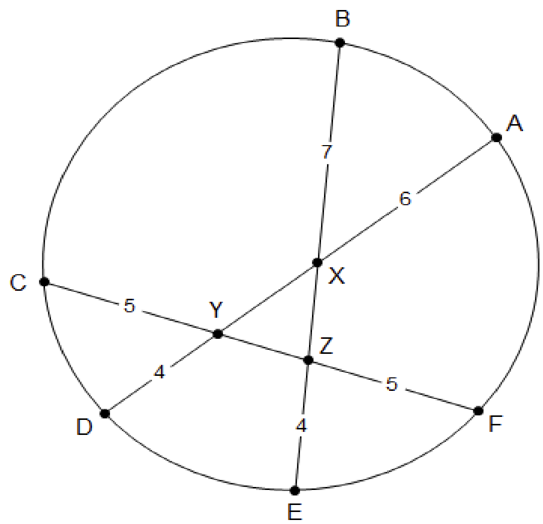
5. **Problem.** Let $a = 2^{2^{10}}$, $b = 125!$, and $c = 3^{3^{6}}$. Which of the following is true?

   (A) $a > b > c$ (B) $a > c > b$ (C) $b > a > c$ (D) $b > c > a$ (E) $c > a > b$

6. **Problem.** What is the coefficient of x^{45} in the polynomial $(x + 20)^{24}(x - 20)^{25}$?

   (A) $45{\cdot}20^5$ (B) $66{\cdot}20^4$ (C) $66{\cdot}20^5$ (D) $276{\cdot}20^2$ (E) $276{\cdot}20^4$

7. **Problem.** Points A, B, C, D, E, and F are placed on a circle; chords AD, BE, and CF intersect in points X, Y, and Z, as shown in the diagram below, with AX=6, BX=7, CY=5, DY=4, EZ=4, and FZ=5.



   What is the perimeter of triangle XYZ?

   (A) $\frac{20}{3}$ (B) $\frac{27}{4}$ (C) $\frac{50}{7}$ (D) $\frac{36}{5}$ (E) $\frac{15}{2}$

8. **Problem.** If you leave a Poisson ® brand light bulb on continuously for 10 days, the probability is 19% that it burns out. If you leave two Poisson ® brand light bulbs on continuously for 15 days, what is the probability that at least one of them burns out, rounded to the nearest percentage point?

   Assume that the probability that a light bulb burns out in any time interval depends only on the length of the time interval, not on how long the light bulb has been left on before then.

   (A) 42% (B) 47% (C) 52% (D) 57% (E) 67%

9. **Problem.** The equation $log_{1/x} 3 - log_{1/4} x = 1$ has two positive real solutions. What is their product?

   (A) 3/4 (B) 1 (C) 4/3 (D) 3 (E) 4

10. **Problem.** Elle's teacher wrote the value of 11^{11} on the board, but Elle copied it down wrong: she wrote down "285,311,760,611", but accidentally swapped two consecutive digits. Which of the following is the true value of 11^{11}?

    (A) 283,511,760,611
    (B) 285,311,670,611
    (C) 285,311,760,161
    (D) 285,311,766,011
    (E) 285,317,160,611

11. **Problem.** An operation ⊛ is defined on the positive real numbers by the formula

    $$x \circledast y = \frac{Cxy}{x+y},$$

    where *C*>0, but the value of *C* is unknown. We say that ⊛ is **commutative** if $x⊛y = y⊛x$ for all positive real numbers *x* and *y*; we say that ⊛ is **associative** if $x⊛(y⊛z) = (x⊛y)⊛z$ for all positive real numbers *x*, *y*, and *z*. Which of the following is true?

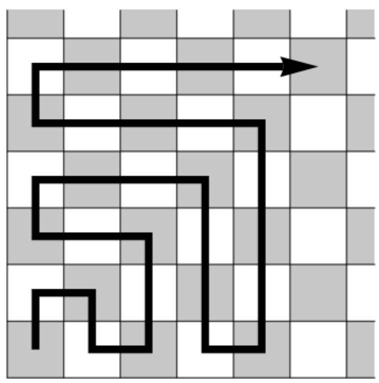
    (A) Without knowing the value of *C*, we cannot say whether ⊛ is associative or commutative.
    (B) ⊛ is associative no matter what *C* is, but without knowing the value of *C*, we cannot say whether ⊛ is commutative.
    (C) ⊛ is commutative no matter what *C* is, but without knowing the value of *C*, we cannot say whether ⊛ is associative.
    (D) ⊛ is both associative and commutative no matter what *C* is.
    (E) None of the above.

12. **Problem.** Carmen lives in a small town and walks to school each day. As she walks, she sees a few cars drive by in both directions, all going exactly at the speed limit. On average, about the same number of cars travel each way along the street Carmen takes to school. However, because Carmen is not standing still, she sees more cars pass her going in the opposite direction.

    In fact, Carmen has collected statistics, and she knows that on an average day, about 5 cars pass her going in the same direction, and about 6 cars pass her going in the opposite direction. What can we conclude about Carmen's walking speed?

    (A) It is less than 10% of the speed of a car.
    (B) It is between 10% and 20% of the speed of a car.
    (C) It is between 20% and 30% of the speed of a car
    (D) It is more than 30% of the speed of a car.
    (E) Cannot be determined.

13. **Problem.** An ant carrying a sack of sand walks through the cells of a 100 by 100 grid, which is colored in a checkerboard pattern. The ant zigzags as shown below:



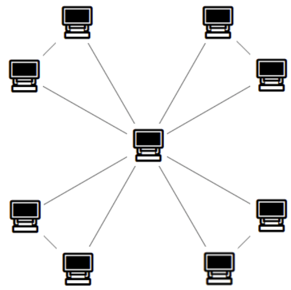
    The ant deposits 1 grain of sand in the first square it visits, 2 grains in the second square, and so on, leaving 10000 grains of sand in the last square. Then, a grain of sand is randomly chosen, so that each of the 1+2+3+...+10000 grains of sand is equally likely. What is the probability that the chosen grain of sand is on a dark-colored square?

    (A) 5001/10000 (B) 4999/10000 (C) 4999/9999 (D) 5001/10001 (E) 5000/10001

14. **Problem.** At a very small middle school, there are *n* 6th graders and *n* 7th graders. Every 6th grader is friends with exactly six 7th graders. Every 7th grader is friends with at least one 6th grader, and no two 7th graders have the same number of friends among the 6th graders. What is the value of *n*?

    (A) 11 (B) 12 (C) 13 (D) 14 (E) 15

15. **Problem.** The diagram below shows connections between several computers in a network; every computer can send messages to any other computer, possibly relayed through some other computers.



For every connection in the network, a coin is flipped; if it lands heads, nothing is changed, but if it lands tails, the connection is removed. What is the probability that after this is done, every computer can still send messages to any other computer?

(A) 1/4096 (B) 1/16 (C) 3/32 (D) 81/256 (E) 91/128

16. **Problem.** The expression $(1+\sqrt{6})^{2024} + (1-\sqrt{6})^{2024}$ simplifies to an integer. What is the last digit of that integer?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

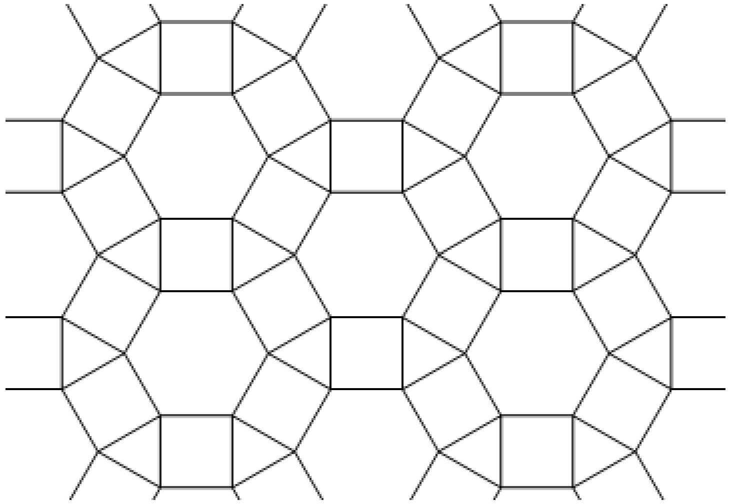
17. **Problem.** A fair 6-sided die has 2 red faces and 4 blue faces. Ray and Bea play a game. Ray rolls the die repeatedly until one of two things happens:

1. The die lands on a red face twice, not necessarily consecutively, or
2. The die lands on a blue face three times, not necessarily consecutively.

Ray wins if (1) happens first, and Bea wins if (2) happens first. What is the probability that Ray wins?

(A) 1/9 (B) 11/27 (C) 5/9 (D) 2/3 (E) 19/27

18. **Problem.** The floor in a very large room is tiled with many hexagonal, triangular, and square tiles in a regular pattern; a fragment of the tiling pattern is shown below.



Approximately what fraction of the floor's area is covered by the square tiles?

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{3}-1}{2}$ (C) $1-\frac{\sqrt{3}}{3}$ (D) $2\sqrt{3}-3$ (E) $\frac{1}{2}$

19. **Problem.** Rina, Rohan, and Ryan go running in the same park at the same time, but not every day. Rina goes running two days in a row, then takes a break for three days. Rohan goes running two days in a row, then takes a break for four days. Finally, Ryan goes running two days in a row, then takes a break for five days.

If Rina, Rohan, and Ryan all go running on March 1st, then all go running again on March 2nd, what is the next day that all three of them will be at the park together?

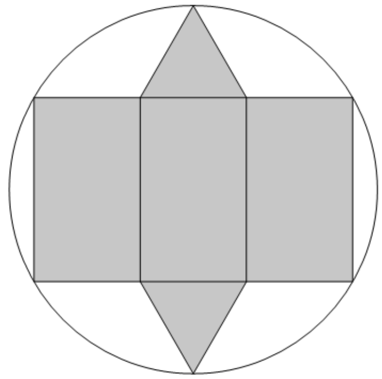
(A) March 31st (B) April 6th (C) April 29th (D) May 25th (E) July 5th

20. **Problem.** For a positive integer n, let $f(n)$ be equal to $n/2$ if n is even, and $3n+1$ if n is odd. A famous open problem called the Collatz conjecture says that, starting from any positive integer n, repeatedly applying f will eventually lead to 1.

How many positive integers n are there for which $f(f(f(f(f(f(f(n)))))))=1$? Here, f is applied to n exactly 7 times.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

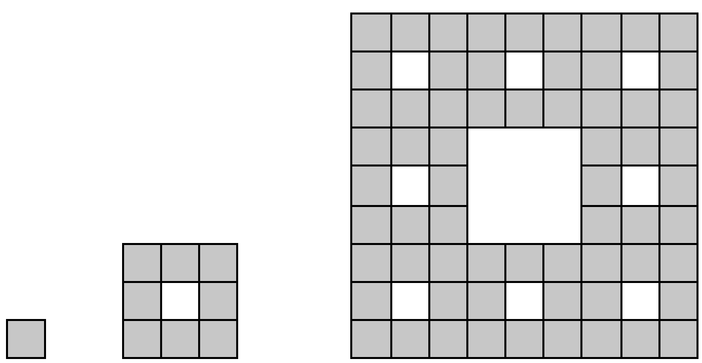
21. **Problem.** Starting from a circular piece of paper of radius 6, Troy cuts out the shape shown below:



Then, Troy folds the shape into a right prism whose base is an equilateral triangle. What is the volume of the resulting prism?

(A) $20\sqrt{2}$ (B) 30 (C) $18\sqrt{3}$ (D) 32 (E) $20\sqrt{3}$

22. **Problem.** A fractal shape is built by starting with a 1×1 square in the first stage. In every stage after that, 8 copies of the previous stage are arranged in a square with the center missing. The first few stages are shown in the illustration below:



Let a_n be the area of the n^{th} stage of the fractal, and let p_n be the perimeter of the n^{th} stage; for example, $a_1 = 1$ and $p_1 = 4$. As n increases, the ratio $a_n : p_n$ gets closer and closer to which of the following?

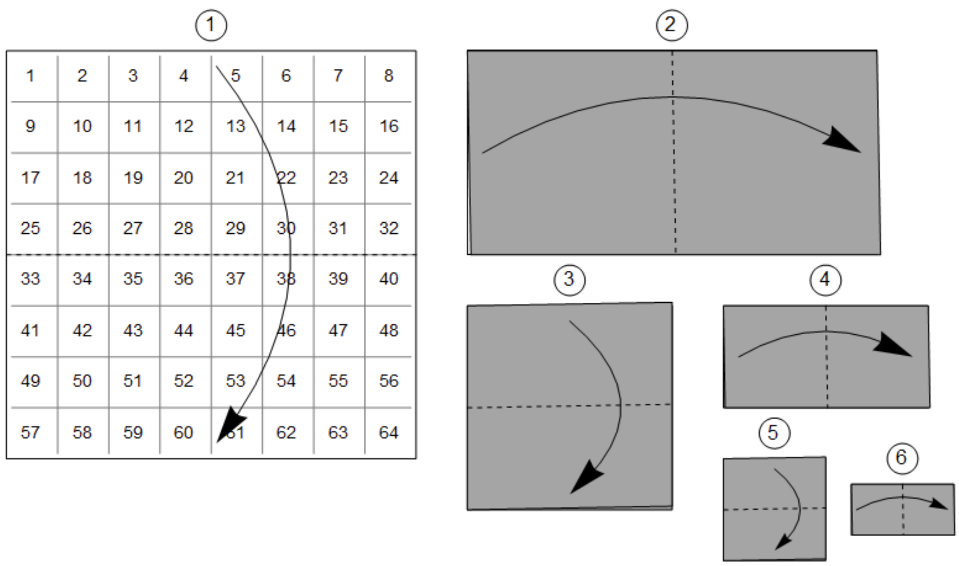
(A) 2 : 1 (B) 5 : 4 (C) 1 : 1 (D) 3 : 4 (E) 2 : 5

23. **Problem.** A **derangement** of a word is an anagram of that word in which no letter is in the correct place; for example, “malem” is a derangement of “lemma”, but “ammel” is not.

If the word “solve” has 44 derangements, the word “fathom” has 265 derangements, and the word “unravel” has 1854 derangements, how many derangements does the word “theorem” have?

(A) 640 (B) 907 (C) 1280 (D) 1814 (E) 1854

24. **Problem.** A square sheet of paper is divided into an 8 by 8 grid whose entries are numbered from 1 to 64, in reading order. Then, the paper is folded 6 times, alternating horizontal folds downward and vertical folds to the right, until it is a 1 by 1 square, as shown below.



The resulting folded paper has 64 layers, and each layer contains one of the numbers from 1 to 64 due to the way the grid was originally numbered. Which of the following numbers is closest to the top layer of the folded paper?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

25. **Problem.** Standing at the point (0,0), Marco takes a step that's 1 unit long, then a step that's 2 units long, then a step that's 3 units long. For each step, Marco randomly picks a direction: up, down, left, or right. How many different points can Marco reach?

(A) 24 (B) 25 (C) 48 (D) 49 (E) 64

Solutions

1. **Solution.** The correct answer is (D) 336.

   Every branch has 3×6+3 = 21 leaves on it. Every bough has 2×21 + 2×6 + 2 = 56 leaves on it. Therefore, the tree has 6×56 = 336 leaves.

2. **Solution.** The correct answer is (C) 20% more.

   Let's say that Aaron paid x for flour and y for sugar. Then Barbara paid only $x/2$ for flour, but $2y$ for sugar; we are given that $x+y=1.25(x/2+2y)$. Solving for x, we get $x=4y$.

   Cece paid x for flour and $2y$ for sugar, so the ratio of Cece's cost over Aaron's cost is $(x+2y)/(x+y)$, or $(4y+2y)/(4y+y) = 1.2$; therefore, Cece paid 20% more than Aaron.

3. **Solution.** The correct answer is (A) $\frac{1}{4}$.

   Method 1: We could, of course, solve the problem by rewriting the fraction as $\frac{1332}{5328}$ and then simplifying. But there is an easier way. In both sums, each digit appears twice in the hundreds place, twice in the tens place, and twice in the ones place, so altogether it is multiplied by 200+20+2. Therefore the numerator is (1+2+3)(200+20+2), the denominator is (7+8+9)(200+20+2), and the factor of 200+20+2 cancels: the fraction can be simplified to $\frac{1+2+3}{7+8+9}$ or $\frac{1}{4}$.

   Method 2: The numbers can be paired up like 123+321=444, 132+312=444, ... ,89+987 = 1776,798+978=1776, ... So, the answer is (3 * 444) / (3 * 1776) = 1/4.

4. **Solution.** The correct answer is (A) Peter.

   The most direct answer is to observe that Reese *must* be telling the truth: otherwise, Reese would be the one who took the cookies, which would still make "Sofia didn't take the cookies" true. Therefore Sofia didn't take the cookies, and since Reese told the truth, Reese also didn't take the cookies.

   Since Sofia didn't take the cookies, her statement is also true, which means Quinn is also innocent, leaving Peter as the only possible culprit.

5. **Solution.** The correct answer is (E) $c > a > b$.

   To compare a and b, write a as 2^{1024}; this is more than $2^{1001} = (2^7)^{143} = 128^{143}$. On the other hand, $b = 125 \cdot 124 \cdot 123 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$; this is the product of 125 factors, all at most 125, so $b < 125^{125}$. Since $128^{143} > 125^{143} > 125^{125}$, $a > b$.

   To compare a and c, write a as 2^{1024}; this is less than $2^{1050} = (2^3)^{350} = 8^{350}$. On the other hand, c can be written as 3^{729}; this is more than $3^{700} = (3^2)^{350} = 9^{350}$. Since $8^{350} < 9^{350}$, $a < c$.

6. **Solution.** The correct answer is (E) $276 \cdot 20^4$.

   Since $(x + 20)(x - 20) = x^2 - 20^2$, we can rewrite the polynomial as $(x^2 - 20^2)^{24}(x - 20)$. In the first factor, $(x^2 - 20^2)^{24}$, there are only even powers of x, so we need to take an x from the final factor of $(x - 20)$, and an x^{44} from $(x^2 - 20^2)^{24}$. This comes from the term with $(x^2)^{22}(-20^2)^2$, whose coefficient is $\frac{24 \cdot 23}{2} = 276$ by the binomial theorem.

7. **Solution.** The correct answer is (D) $\frac{36}{5}$.

   When the two chords AD and BE intersect at point X, the products AX·XD and BX·XE must be equal; in other words, 6(XY+4) = 7(XZ+4). Points Y and Z give us similar equations: 5(YZ+5)=4(XY+6) and 4(XZ+7) = 5(YZ+5).

   If we spot that 5(YZ+5) occurs twice, we can solve the system more quickly: this tells us that 4(XY+6) = 4(XZ+7), which simplifies to XZ = XY-1, so in the first equation, 6(XY+4) = 7(XY+3). This means that XY=3, so XZ=2; now 5(YZ+5) = 4(XY+6) = 36, which means that YZ = $\frac{11}{5}$, and the perimeter of triangle XYZ is XY + XZ + YZ = $\frac{36}{5}$.

8. **Solution.** The correct answer is (B) 47%.

   Let p be the probability that a light bulb lasts an entire 5-day interval without burning out. Then the probability that it lasts 10 days is p^2, so we are given that $1 - p^2 = 0.19$. Therefore $p^2 = 0.81$, and $p = 0.9$.

   The probability that a light bulb lasts 15 days is p^3, and the probability that both light bulbs last 15 days is p^6, so we are looking for the quantity $1 - p^6$. Computed exactly, this is 0.468559, but we can also find the right answer by approximating. Begin with $p^3 = 0.729$, which is between 0.7 and 0.75. We can compute $1 - 0.7^2 = 0.51$ and $1 - 0.75^2 = 7/16 = 0.4375$, so we know that the rounded value of $1 - p^6$ will be between 44% and 51%. This can only be (B).

9. **Solution.** The correct answer is (E) 4.

We can rewrite $log_{1/x}\ 3$ as $-\frac{log\ 3}{log\ x}$ and $log_{1/4}\ x$ as $-\frac{log\ x}{log\ 4}$, where the base of the logarithm can be anything we like. Therefore, in terms of y = log x, the equation becomes a quadratic equation:

$-\frac{log\ 3}{y}+\frac{y}{log\ 4}=1$ or y^2 - y log 4 - log 3 log 4 = 0.

This equation has two solutions y_1 and y_2 whose sum, by Vieta's formula, is log 4. If y_1 = log x_1 and y_2 = log x_2, then log(x_1x_2) = log x_1 + log x_2 = y_1 + y_2 = log 4, so x_1x_2 = 4.

10. **Solution.** The correct answer is (B) 285,311,670,611.

We can use the divisibility rule for 11 to determine where the error was made. For a number divisible by 11, if we take the alternating sum of its digits, the result should also be divisible by 11; however, if we compute 2+8-5+3-1+1-7+6-0+6-1+1, the result is 13, which is not divisible by 11.

If we swap two consecutive digits whose difference is d, the alternating sum changes by ±2d. The only way to change 13 to a multiple of 11 by adding or subtracting 2d, where d<10, is to subtract 2. Therefore d = 1: Elle's error was swapping two consecutive digits whose difference was 1. The only such pair in 285,311,760,611 is the pair (7,6), so they should be swapped, yielding the true value of 11^{11}: 285,311,670,611.

11. **Solution.** The correct answer is (C).

No matter what the value of C is, we can say that $\frac{Cxy}{x+y}=\frac{Cyx}{y+x}$, making ⊛ commutative. But we cannot be certain whether ⊛ is associative or not:

- On one hand, if C=1, then we can rewrite x⊛y as $\frac{1}{1/x+1/y}$, and both x⊛(y⊛z) and (x⊛y)⊛z are equal to $\frac{1}{1/x+1/y+1/z}$, so in this case, ⊛ is associative.
- On the other hand, 1⊛(1⊛2) simplifies to $\frac{2C^2}{2C+3}$, while (1⊛1)⊛2 simplifies to $\frac{2C^2}{C+4}$, which are not equal unless 2C+3 = C+4, or C=1. So for most values of C, the operation ⊛ is not associative.

Therefore (C) is the only correct statement about ⊛.

12. **Solution.** The correct answer is (A) It is less than 10% of the speed of a car.

Let's suppose that Carmen starts walking at time 0 and ends at time 1, and a car is k times faster than Carmen. Then a car going the same way as Carmen will pass her if it passes by Carmen's house between time 0 and time 1 - 1/k; any later than that, and Carmen will reach school first. On the other hand, a car going the opposite way will pass Carmen if it passes by Carmen's school between time -1/k and 1; any earlier than that, and it will reach Carmen's house before she's left.

Therefore Carmen must be seeing (1+1/k)/(1 - 1/k) = (k+1)/(k-1) more cars going the opposite direction. Setting (k+1)/(k-1) = 6/5 and solving, we get k = 11. Therefore a car is 11 times faster than Carmen, making Carmen's walking speed less than 10% of the speed of a car.

13. **Solution.** The correct answer is (E) 5000/10001.

The exact zigzagging pattern does not matter; as long as the ant visits all 10000 squares by moving from one square to an adjacent square, it will leave an odd number of grains on each dark-colored square and an even number of grains on each light-colored square.

Therefore there are 1+3+5+...+9999 grains of sand on dark-colored squares, out of 1+2+3+...+10000 total. The sum of odd numbers has a very nice pattern to it: $1+3=2^2$, $1+3+5=3^2$, $1+3+5+7=4^2$, and so on, with $1+3+5+...+9999=5000^2$. The sum of even numbers is 5000 more than the sum of odd numbers, since each even number is 1 more than the corresponding odd number. Therefore the probability we want is
$\frac{5000^2}{5000^2+(5000^2+5000)} = \frac{5000}{5000+(5000+1)} = \frac{5000}{10001}$.

14. **Solution.** The correct answer is (A) 11.

There are n 7th graders, each with between 1 and n friends among the 6th graders, and each with a different number of friends. Therefore there must be one 7th grader with 1 friend, one 7th grader with 2 friends, one 7th grader with 3 friends, and so on through one 7th grader with n friends (among the 6th graders). The total number of friendships from 7th grade to 6th grade is therefore $1 + 2 + 3 + \ldots + n = n(n+1)/2$.

On the other hand, the total number of friendships from 6th grade to 7th grade is $6n$, because each of the n 6th graders has 6 friends in 7th grade. These must be equal, because we are counting the friendships between the two grades in two ways. So $6n = n(n+1)/2$, or $6 = (n+1)/2$, which means $n+1 = 12$, or $n=11$.

15. **Solution.** The correct answer is (B) 1/16.

In each triangle of computers, there is a ½ probability that at least two connections are removed, and a ½ probability that at most one connection is removed.

If none of the triangles lost more than one connection (which happens with probability $(½)^4$, or 1/16), then each triangle can still communicate to the central computer, and therefore all the computers can still send messages to each other.

If, on the other hand, there is a triangle that lost two or more connections, then not all computers can communicate: either one of the outer computers in that triangle cannot send messages to anywhere, or else the two outer computers in that triangle can send messages to each other, but nowhere else. So the scenario in the previous paragraph, which happens with probability 1/16, is the only way that all computers can communicate with all others.

16. **Solution.** The correct answer is (D) in the case of 2024

Method I: One of the ways to solve the problem is to observe the repetition in the last digit of $(1+\sqrt{6})^n + (1-\sqrt{6})^n$. Writing $(1+\sqrt{6})^n$ as $a + b\sqrt{6}$, we will get $(1+\sqrt{6})^{n+1} = (a+6b) + (a+b)\sqrt{6}$, and we can keep track of the units digit of a and b: starting from (1,1) when n=0, they go to (7,2), then to (9,9), then to (3,8), then back to (1,1). If $(1+\sqrt{6})^n = a + b\sqrt{6}$, then $(1+\sqrt{6})^n + (1-\sqrt{6})^n = 2a$, which will loop through the values 2, 4, 8, 6, 2, 4, 8, 6, … as n increases.

Method II: An alternate solution is to observe that for even k, $(\sqrt{6})^k$ and 6^k have the same last digit: 6. In the simplification of $(1+\sqrt{6})^n + (1-\sqrt{6})^n$, all the odd powers of √6 will cancel, leaving only the even powers, and so we should get the same answer if we replace √6 by 6. Now we must find the last digit of $(1+6)^n + (1-6)^n = 7^n + (-5)^n$, which is a more familiar problem. The powers of 7 end in the digits 7, 9, 3, 1, 7, 9, 3, 1… repeating every 4 steps, and the powers of -5 always end in 5 (added or subtracted), which changes this sequence to 2, 4, 8, 6, 2, 4, 8, 6, …, just as in the previous solution.

17. **Solution.** The correct answer is (B) 11/27.

We can build a table of the probability that Ray wins given that r red faces and b blue faces have already come up:

	b=0	b=1	b=2	b=3
r=0	11/27	7/27	1/9	0
r=1	19/27	5/9	1/3	0
r=2	1	1	1	

The table is filled in by the following rule: first, each entry in the $r=2$ row is 1 (because Ray has already won) and each entry in the $b=3$ row is 0 (because Bea has already won). Every other entry is ⅓ times the entry below it, plus ⅔ times the entry to its right, because there's a ⅓ chance of another red face and a ⅔ chance of another blue face coming up on the next roll.

The entries can be computed in reverse order, and the $r=0$, $b=0$ entry gives us the probability that Ray wins at the beginning of the game.

18. **Solution**. The correct answer is (D) $2\sqrt{3}-3$.

There are two factors to consider. The first is the area of each shape: if each segment has length 1, then a square has area 1, a triangle has area $\sqrt{3}/4$, and a hexagon has area $3\sqrt{3}/2$. The second factor is how often each shape occurs. Each triangle borders three squares, but each square only borders two triangles; therefore for N squares there are only 2N/3 triangles. Each hexagon borders six squares, but each square only borders two hexagons; therefore for N squares there are only N/3 hexagons.

Putting this together, if there are N squares of area 1, then the total area of the squares is N, the total area of the triangles is $N\sqrt{3}/6$, and the total area of the hexagons is $N\sqrt{3}/2$; the fraction of the area covered by squares is

$$\frac{N}{N + N\sqrt{3}/6 + N\sqrt{3}/2} = \frac{1}{1+\sqrt{3}/6 + \sqrt{3}/2} = \frac{3}{3+2\sqrt{3}} = 2\sqrt{3}-3,$$

the answer given in (D). (The question asks for an approximate ratio, because this answer is only an approximation for any room of finite size, due to the effects at the edge of the room.)

19. **Solution.** The correct answer is (B) April 6th.
If we number the days starting from 0 (March 1st), then Rina is at the park on every day of the form $5a$ or $5a+1$, Rohan is at the park on every day of the form $6b$ or $6b+1$, and Ryan is at the park on every day of the form $7c$ or $7c+1$. There are 8 such days every 210 days (where 210 is the LCM of 5, 6, and 7). They are:

- Day 0 and 1, which we are given.
- Day 120, the first multiple of 30 which is one more than a multiple of 7.
- Day 91, the first multiple of 7 which is one more than a multiple of 30.
- Day 175, the first multiple of 35 which is one more than a multiple of 6.
- Day 36, the first multiple of 6 which is one more than a multiple of 35.
- Day 126, the first multiple of 42 which is one more than a multiple of 5.
- Day 85, the first multiple of 5 which is one more than a multiple of 42.

Therefore day 36 (which is April 6th) is the next day all three of them are at the park.

20. **Solution.** The correct answer is (C) 6.

We work backwards, undoing the operation defined by f in every way possible:

1. The only number n such that $f(n)=1$ is $n=2$.
2. The only number n such that $f(n)=2$ is $n=4$.
3. There are two numbers n such that $f(n)=4$: $n=1$ (because 1 is odd and $3(1)+1=4$) and $n=8$. We will ignore $n=1$ for now.
4. The only number n such that $f(n)=8$ is $n=16$.
5. There are two numbers n such that $f(n)=16$: $n=5$ (because 5 is odd and $3(5)+1=16$) and $n=32$.
6. There are two numbers n such that $f(n)=5$ or $f(n)=32$: $n=10$ and $n=64$.
7. There are four numbers n such that $f(n)=10$ or $f(n)=64$: $n=3$, $n=20$, $n=21$, and and $n=128$.

All four starting values of 3, 20, 21, and 128 lead to 1 after 7 steps *for the first time*. Because we discarded the case $f(f(f(1)))=1$ in our analysis, we must also include 16 (which leads to 1 after 4 steps, and again after 7 steps) and 2 (which leads to 1 after 1 step, and again after 4 steps, and again after 7 steps), for a total of 6 solutions.

21. **Solution.** The correct answer is (C) $18\sqrt{3}$.

Let A, B, C, D, E, F be the six points where the shape Troy cuts out touches the circle, starting at the top and going counterclockwise. Let P, Q, R, S be the corners of the middle rectangle, such that P and Q lie on BF and R and S lie on CE, with P to the left of Q and S to the left of R.

We know that triangles APQ and DRS are equilateral triangles; in order to obtain a prism as the final result, we also want BP = PQ = QF and ER = RS = SC. Therefore triangles ABP, CDS, DER, and EFQ are all isosceles triangles; their vertex angles are 120 degrees, so their other angles are all 30 degrees. This tells us that ABCDEF is a regular hexagon: it is inscribed in a circle, and all its angles are 120 degrees.

If O is the center of the circle, then triangle BCO is equilateral; BO = CO = 6, the radius of the circle, so BC = 6 as well. This tells us the height of the prism. Triangle BCF is a right triangle with BC = 6 and CF = 12, so BF = $6\sqrt{3}$ and BP = PQ = QF = $2\sqrt{3}$. This means that triangle APQ is equilateral with side $2\sqrt{3}$, so its height is 3 and its area is $3\sqrt{3}$. Finally, the volume of the prism is the height times the area of the base, or $18\sqrt{3}$.

22. **Solution.** The correct answer is (B) 5 : 4.

The value of a_n is simply 8^{n-1}: at each stage, the area is multiplied by 8, since we take 8 copies of the previous stage. The value of p_n is more complicated. There is an outer perimeter of $4 \cdot 3^{n-1}$: the fractal fits inside a 3^{n-1} by 3^{n-1} square at the n^{th} stage. There are also many holes contributing to the perimeter:

- When n is at least 2, there are 8^{n-2} holes which are 1 by 1, with perimeter 4 each.
- When n is at least 3, there are 8^{n-3} holes which are 3 by 3, with perimeter 12 each.
- When n is at least 4, there are 8^{n-4} holes which are 9 by 9, with perimeter 36 each.

The total area contributed by these holes is $4 \cdot 8^{n-2} + 12 \cdot 8^{n-3} + 36 \cdot 8^{n-4} + \ldots$, with n-1 terms in the sum at the n^{th} stage. We can write this sum as $8^{n-1}(1/2 + 3/16 + 9/128 + \ldots)$, where the expression in the parentheses is a finite geometric series with first term 1/2 and common ratio 3/8. As n gets larger and larger, this finite geometric series will get closer and closer to the infinite series, whose value is $\frac{1/2}{1-3/8}$ or 4/5.

Compared to a term with 8^{n-1} in it, the term of $4 \cdot 3^{n-1}$ that arises from the outer perimeter will eventually be negligible. Therefore the ratio $a_n : p_n$ will get closer and closer to 8^{n-1} : $8^{n-1}(4/5)$, which can be simplified to 5 : 4.

23. **Solution.** The correct answer is (A) 640.

To begin with, why is the answer not 1854, just as for “unravel”? It is because “theorem” has a repeated letter in it: “e” occurs twice. If we wrote “theorem” as “theorEm” and treated “e” and “E” as different, then there would be 1854 derangements, but some of them would not be true derangements of “theorem”: those would be the ones in which “e” is in the place of “E”, or “E” is in the place of “e”, or both.

If we swap “e” and “E” in these “false derangements”, then we get the anagrams of “theorEm” where either “e” or “E” is in its correct place, or both, but no other letter is correct. There are 265 anagrams of “theorEm” where only “e” is in the correct place: this is the same as the number of derangements of a 6-letter word with no repeated letters, such as “fathom”. There are also 265 anagrams of “theorEm” where only “E” is in the correct place, and finally there are 44 anagrams of “theorEm” where “e” and “E” are the two letters in the right places.

Excluding all three cases, we get 1854 - 265 - 265 - 44 = 1280 derangements of “theorEm” where, additionally, “e” is not in the place of “E” and “E” is not in the place of “e”. However, this counts each derangement of “theorem” twice: for example, “ethrome” is counted once as “Ethrome” and once as “ethromE”. So we must divide by 2 to get 640, the final answer.

24. **Solution.** The correct answer is (D) 7.

The square originally numbered 64 stays in place as we fold the paper, and we can number the other cells of the grid by the number of the folding step (between 1 and 6) which places them on top of that square:

2	6	6	4	4	6	6	1
5	6	6	5	5	6	6	5
5	6	6	5	5	6	6	5
3	6	6	4	4	6	6	3
3	6	6	4	4	6	6	3
5	6	6	5	5	6	6	5
5	6	6	5	5	6	6	5
2	6	6	4	4	6	6	

The numbers 1, 3, 5, 7, 9 end up placed on top of the bottom right square in steps 2, 6, 4, 6, 5, respectively. So 3 and 7 (the numbers that are placed on top of the bottom right square in step 6) are closer to the top of the grid.

To decide between 3 and 7, we watch what happens to them closely. After fold 2, they end up above square 63 in the grid, with 3 above 7. Then, they stay put until fold 6, which puts them both above square 64, and also reverses their order. Therefore 7 is above 3 in the stack.

25. **Solution.** The correct answer is (D) 49.

There are two things we can guarantee about Marco's final location (x,y). First, $|x|+|y|$ can be at most 6, because that's the maximum total change in coordinates. Second, the sum $x+y$ must be even, because it starts even, the first and third step change it by an odd number, and the second step changes it by an even number. The points that satisfy both conditions are arranged in a 7 by 7 diamond with vertices at (0,6), (6,0), (0,-6), and (-6,0); there are 49 of them.

To be sure that this is the correct answer, we can check that every point in this diamond can be reached, but by symmetry, we only need to check the points where $0 \leq x \leq y$. There are 10 points to check: (0,0), (0,2), (0,4), (0,6), (1,1), (1,3), (1,5), (2,2), (2,4), and (3,3). We can reach:

- (0,0) by going up, up, and down;
- (0,2) by going up, down, and up;
- (0,4) by going down, up, and up;
- (0,6) by going up, up, and up;
- (1,1) by going right, down, and up;
- (1,3) by going left, right, and up;
- (1,5) by going right, up, and up;
- (2,2) by going down, right, and up;
- (2,4) by going up, right, and up;
- (3,3) by going right, right, and up.